Killara HIGH SCHOOL

MATHEMATICS DEPARTMENT

Student Name

2022 **YEAR 12**

Mathematics Extension 1

Task 4

Date: 15/08/2022

Q	Marks
MC	/10
11	/15
12	/16
13	/15
14	/14
Total	/70

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 70

Section I - 10 marks

- Allow about 15 minutes for this section

Section II - 60 marks

- Allow about 1 hour 45 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 30% of the course.

Section I

10 marks

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Use the multiple-choice sheet for Question 1–10

---

1 Given $|\underset{\sim}{a}| = 4$, $|\underset{\sim}{b}| = \frac{3}{7}$ and the angle between two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{2\pi}{3}$.

The dot product, $\underset{\sim}{a} \cdot \underset{\sim}{b}$, is

(A) $-\dfrac{7}{6}$

(B) $-\dfrac{6}{7}$

(C) $\dfrac{6}{7}$

(D) $\dfrac{7}{6}$

2 Vector $\underset{\sim}{A}$ has components $A_x > 0$, and $A_y > 0$. The angle that this vector makes with the positive x-axis must be in the range

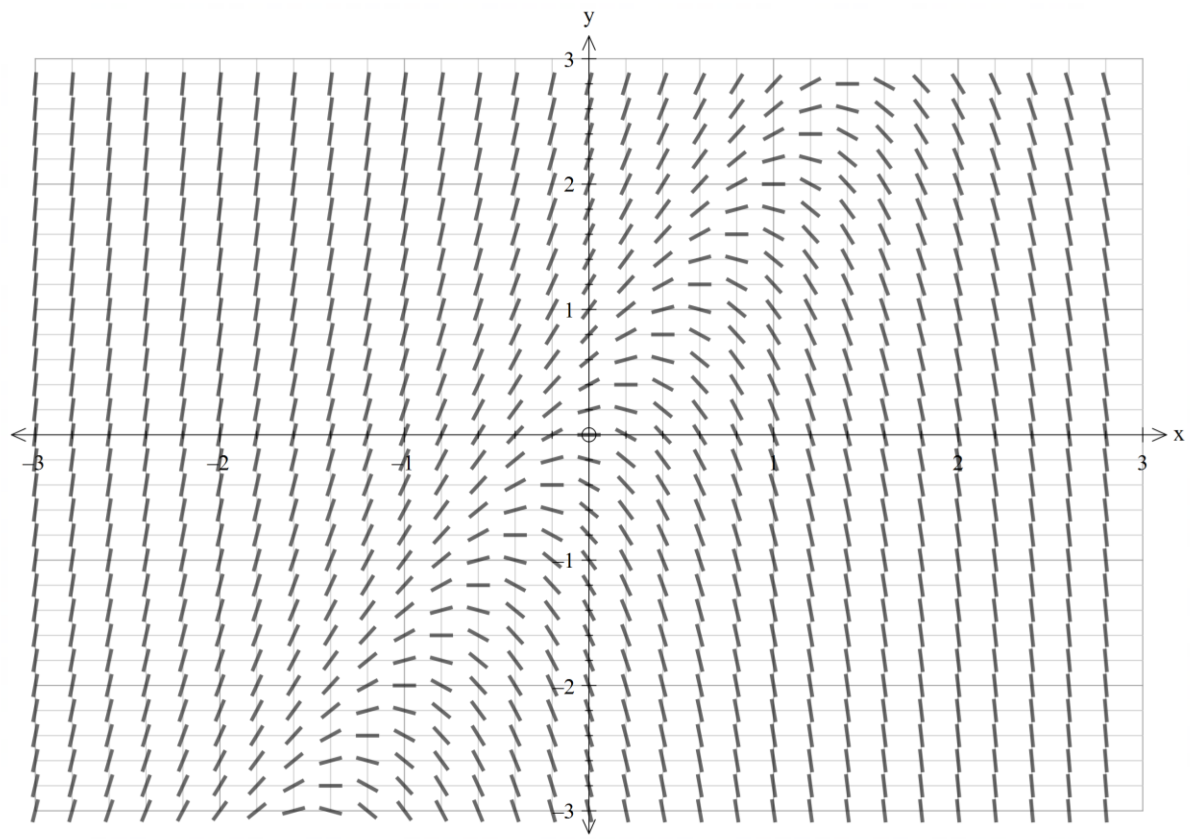
(A) 0° to 90°

(B) 90° to 180°

(C) 180° to 270°

(D) 270° to 360°

3



The differential equation that best represents the above direction field is

(A) $$\frac{dy}{dx} = 2y - 4x$$

(B) $$\frac{dy}{dx} = 4x - 2y$$

(C) $$\frac{dy}{dx} = 2y + 4x$$

(D) $$\frac{dy}{dx} = \frac{1}{2x + 1}$$

Examination continues on next page

4 Find the coefficient of the x^3 term in $(x-3)^{10}$.

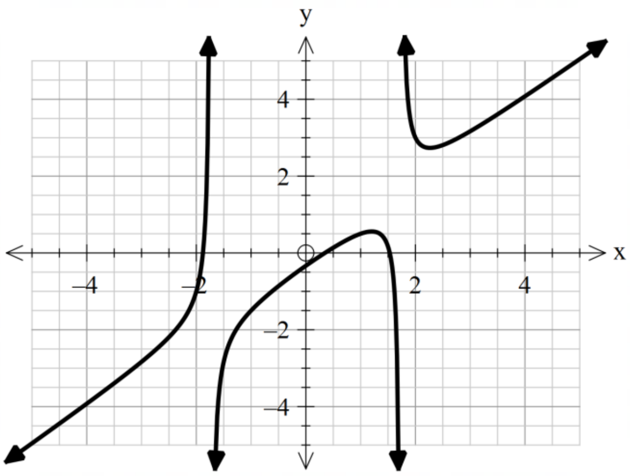
(A) $-3240x^3$

(B) -3240

(C) $-262440x^3$

(D) -262440

5 Below is the graph of $y = f(x)$



Which of the following is a graph of $y^2 = f(x)$

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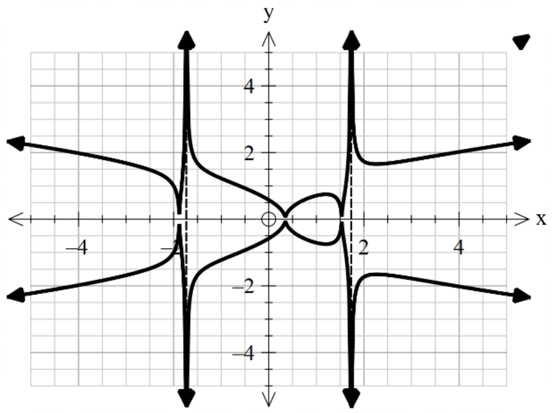
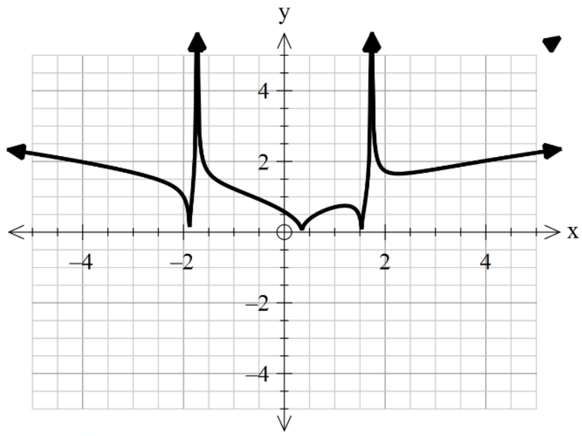
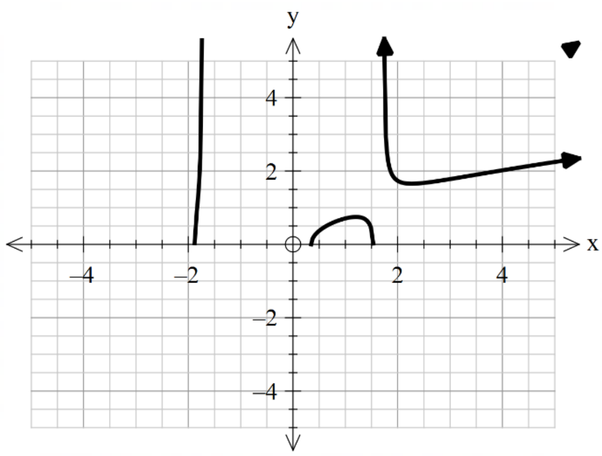


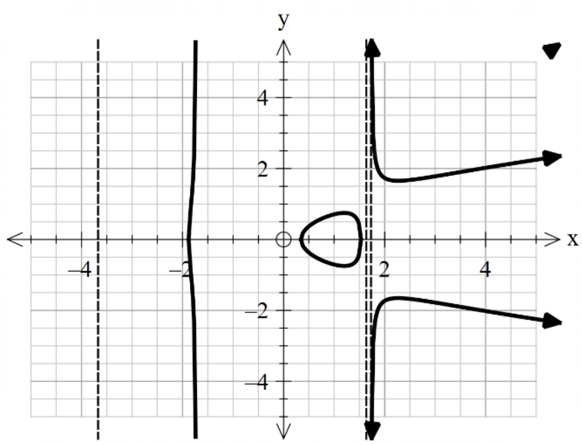
(B)



(C)



(D)



Examination continues on next page

6 Find the cartesian equation for the graph formed by the parametric equations:

$$x = 4 + 4\cos t$$
$$y = 5 + 3\sin t$$

(A) $\dfrac{(x+4)^2}{16} + \dfrac{(y+5)^2}{9} = 1$

(B) $\dfrac{(x-4)^2}{16} + \dfrac{(y-5)^2}{9} = 1$

(C) $(x-4)^2 + (y-5)^2 = 1$

(D) $\dfrac{(x-5)^2}{9} + \dfrac{(y-4)^2}{16} = 1$

7 The general solution of the differential equation $x^3 + y^3\frac{dy}{dx} = 4$ can be shown as:

(A) $x^3 + y^3 = 4x + c$

(B) $x^3 + y^3 = 16x + c$

(C) $x^4 + y^4 = 4x + c$

(D) $x^4 + y^4 = 16x + c$

8 How many solutions are there to the equation $x + y + z = 4$ given that x, y and z are non-negative integers?

(A) 4

(B) 15

(C) 17

(D) 24

9 Given that $f(x) = x^4 + x^2 + 3x - 1$ and $g(x) = x^2 - 1$. If the $f(x) = Q(x)g(x) + R(x)$, which of the following is true?

i) $Q(x) = x^2 + 2$
ii) $R(x) = 3x + 1$
iii) $g(x)$ is a factor of $f(x)$

(A) i) only

(B) ii) only

(C) i) and ii) only

(D) iii)

Examination continues on next page

10 Which of the following is equivalent to $\frac{d}{dx}f^{-1}(x)$, given that $f(x)$ and $f^{-1}(x)$ are continuous differentiable functions?

(A) $\dfrac{1}{f'(f^{-1}(x))}$

(B) $-\dfrac{1}{f^{-2}(x)}$

(C) $f'(x)f(f^{-1}(x))$

(D) $\dfrac{1}{f^{-1}\left(f'(x)\right)}$

End of Section I

Section II

60 marks

Allow about 1 hour 45 minutes for this section

In Questions 11 – 14, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Given $\underset{\sim}{a} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$,

i) Find $\underset{\sim}{a} + \underset{\sim}{b}$ in component form. **1**

ii) Calculate $|-2\underset{\sim}{a} - 7\underset{\sim}{b}|$ **1**

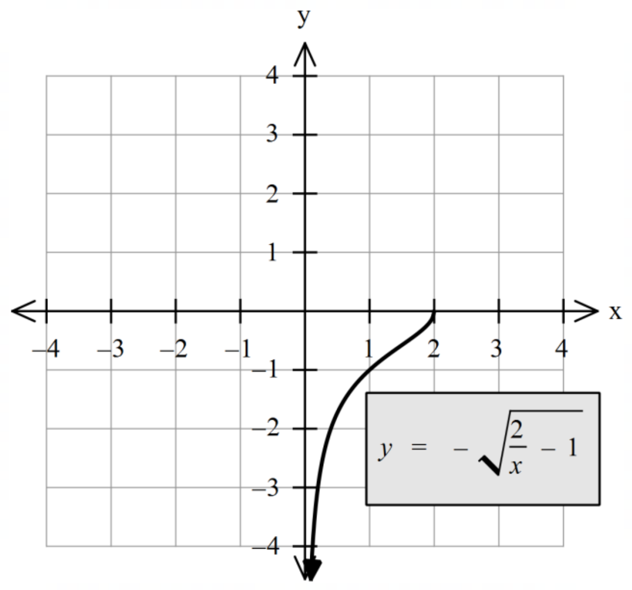
(b) A square metal sheet is being heated such that its side length is increasing at a uniform rate of 0.05cms^{-1}. Find the instantaneous rate of change of its area at the instant when the side length is 3.1cm. **2**

(c) i) Express $2\sqrt{3}\cos x - 2\sin x$ in the form $R\cos(x + \alpha)$ **2**

ii) Hence, solve $2\sin x = 1 + 2\sqrt{3}\cos x$, $0 \leq x \leq \pi$ **1**

Examination continues on next page

(d) The one-to-one function $f(x) = -\sqrt{\frac{2}{x} - 1}$ is graphed below.



(i) State the domain and range of $f(x)$. **1**

(ii) Sketch the graph of $f^{-1}(x)$. **1**

(iii) Find the equation of $f^{-1}(x)$ and state its domain and range. **2**

(e) According to Mars, 24% of M&M plain candies are blue. Assuming the claimed rate of 24% is correct, use the normal approximation of a binomial to find the probability of randomly selecting 100 M&Ms and getting 27 or more that are blue? **2**

(f) $OABC$ is a rhombus, where O is the origin, $\overrightarrow{OA} = \binom{3}{7}$, $\overrightarrow{OC} = \binom{7}{3}$. Find the lengths of the diagonals. **2**

Question 12 (16 marks)

(a) Simplify **2**

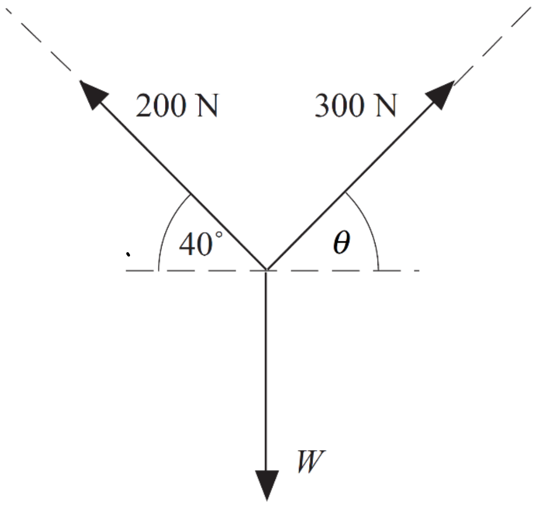
$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}$$

(b) Using the dot product find the angle between the vectors given by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2-\sqrt{3} \\ -1+2\sqrt{3} \end{pmatrix}$. **2**

(c) Using the substitution $u = x^4 + 4x^2$ integrate **3**

$$\int (x^4 + 4x^2 + 5)(x^3 + 2x)\,dx$$

(d) The diagram shows the force of gravity, W, that pulls straight down on an object. Two forces of 200 N and 300 N are exerted by cables as shown.
If the resultant force is zero, find θ and W. **3**



Examination continues on next page

(e) A bowl of water heated to 100 °C is placed in a cool room where the temperature is maintained at $-5\ °C$. After t minutes, the temperature T °C of the water is changing so that $\frac{dT}{dt} = -k(T + 5)$.

i) Show that $T = A\,e^{-kt} - 5$ satisfies this equation, and find the value of A. **2**

ii) After 20 minutes, the temperature of the water has fallen to 40 °C. How long, to the nearest minute will the water need to be in the cool room before ice begins to form? [That is the temperature falls to 0 °C.] **2**

(f) A set of encyclopedias is numbered 1 to 13 is placed randomly on a bookshelf. In how many ways can they be placed if encyclopedias 1 and 2 are in the correct order, but not necessarily next to each other. **2**

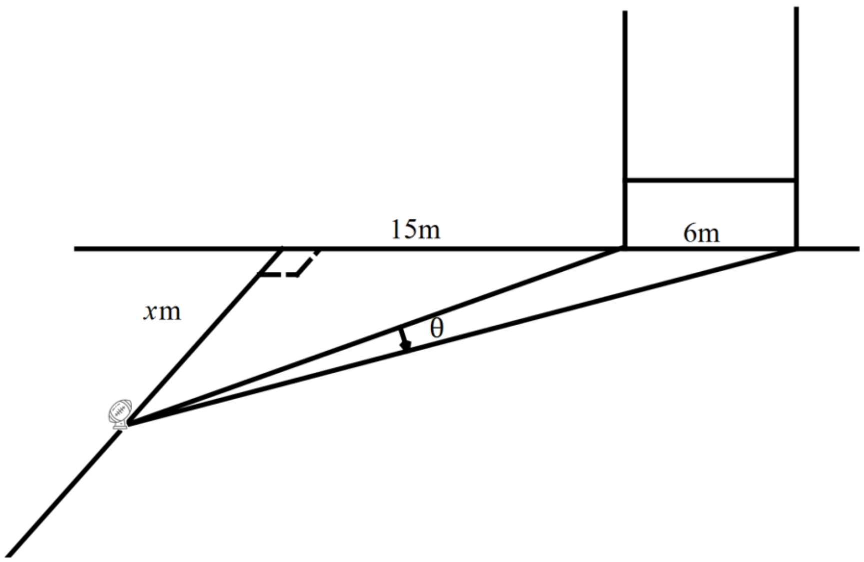
Question 13 (15 marks)

(a) Quality control for the manufacturing of bolts is carried out by taking a random selection of 15 bolts from a batch of 10000. Empirical data has determined that 10% of bolts are defective. If three or more bolts in the sample are found to be defective, that batch is rejected

i) Find the probability that batch is rejected. **2**

ii) The cost to produce the batch of 10000 is \$20. Each batch is then either sold for \$40 or it is sold as scrap metal for \$5. Find the expected value of the batch of 10000. **1**

(b) A rugby player is attempting to kick a ball through the goal posts after his team has scored a try. He can take it anywhere along a line running parallel to the sideline, 15m meters to the side of the goal post. Given that the goal posts are 6m apart:



i) Show that **3**

$$\theta = \tan^{-1}\left(\frac{6x}{315 + x^2}\right)$$

ii) Hence find the distance x metres from the try line that makes the value of θ a maximum. **3**

Examination continues on next page

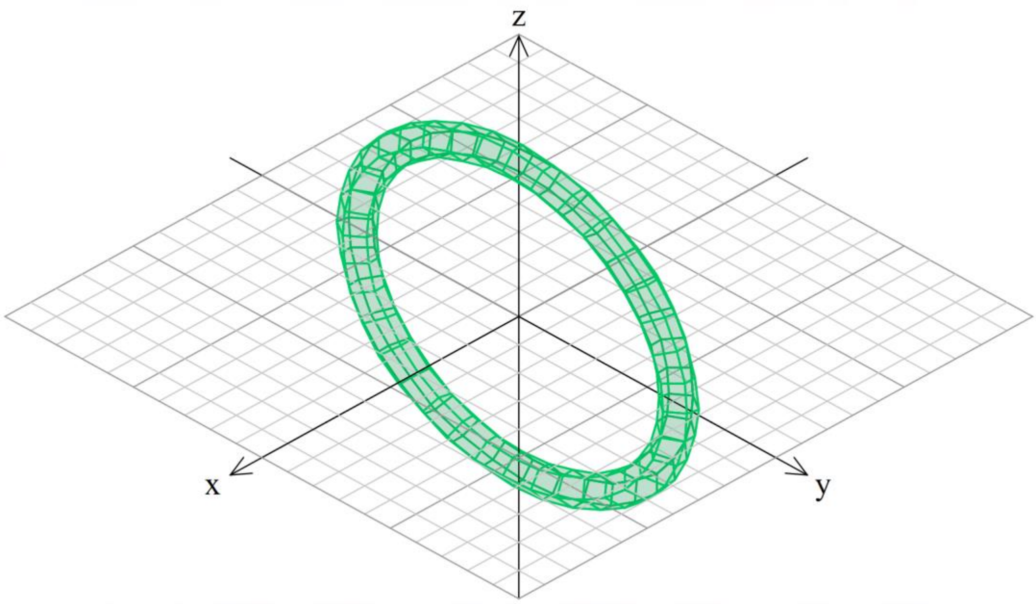
(c) Prove by induction that $n^2 + 2n$ is divisible by 8 for all even integers $n \geq 2$. **3**

(d) For nonzero constants c and d, the equation $4x^3 - 12x^2 + cx + d = 0$ has three real roots and two of those roots add to zero. Find $\frac{c}{d}$. **3**

Question 14 (14 marks)

(a) A donut shaped pool toy is designed by rotating the circle $(y - 125)^2 + x^2 = 144$ around the x-axis. **4**

Find the exact volume of the pool toy.



(b) Gary is training for a triathlon. Over a 30-day period, he pledges to train at least once per day, and 45 times in all. If he sticks to his pledge show that there will be a period of consecutive days where he trains exactly 14 times. **3**
(Hint: Consider the sets $\{S_1, S_2, S_3, \dots, S_{30}\}$ and $\{S_1 + 14, S_2 + 14, S_3 + 14, \dots, S_{30} + 14\}$ where S_i represents the number of training sessions that have been completed by day i.)

Examination continues on next page

(c) A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water.

Let the amount of salt in the tank be represented by S kg and the time be represented by t min.

i) Show that the expression for $\frac{dS}{dt}$ is **1**

$$\frac{dS}{dt} = 0.1 - 0.1S$$

ii) Hence, find an expression for the amount of salt, S, in the tank at any time, t. **3**

(d) By substituting $t = \text{a}\sin(u)$, where a is some real constant, show that **3**

$$\int \sqrt{a^2 - t^2}\,dt = \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + \frac{1}{2}t\sqrt{a^2 - t^2} + c$$

End of Examination

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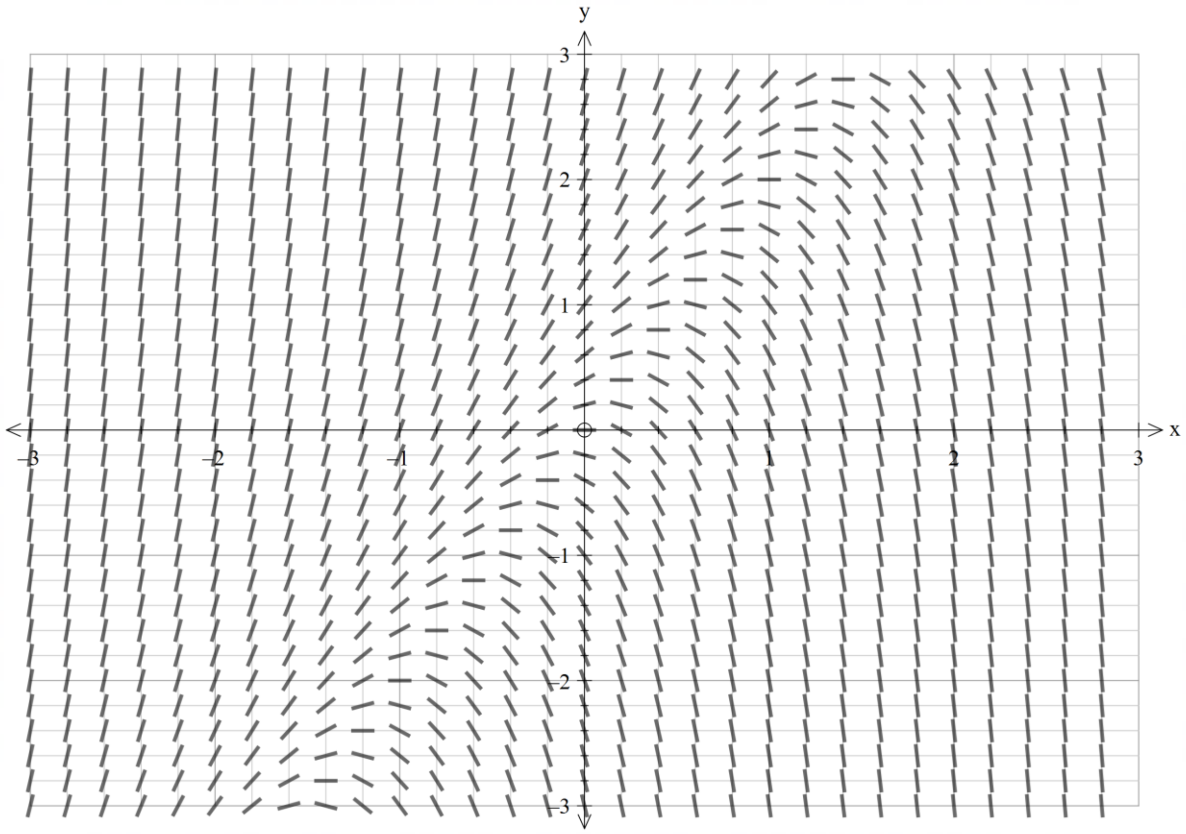

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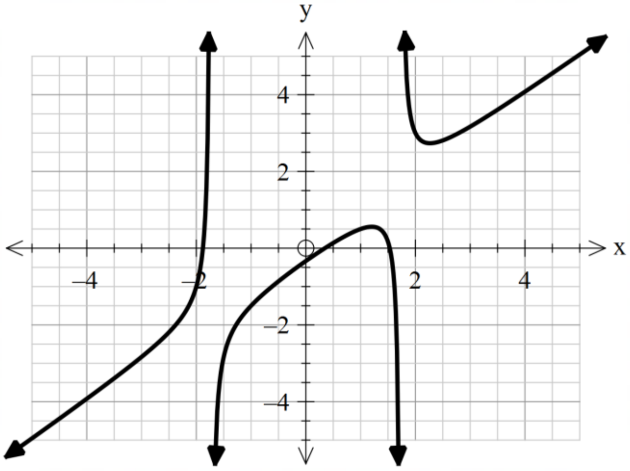

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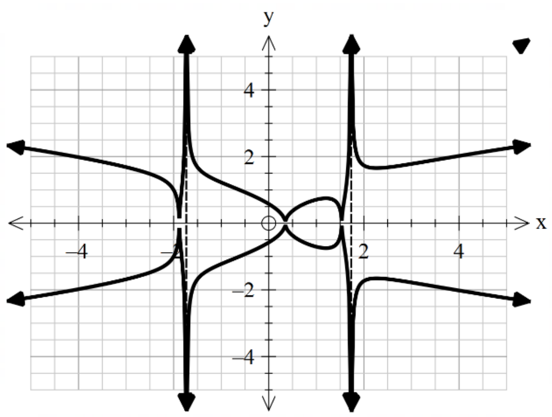

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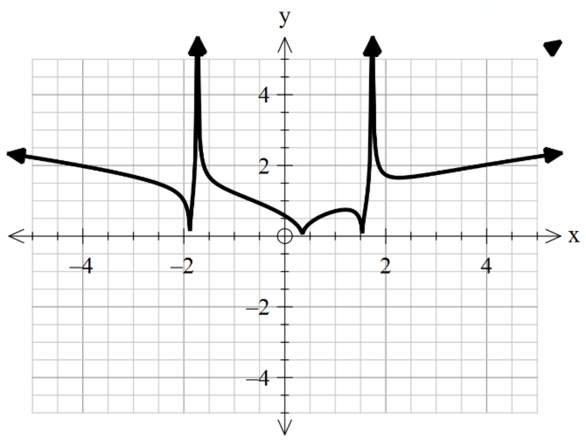
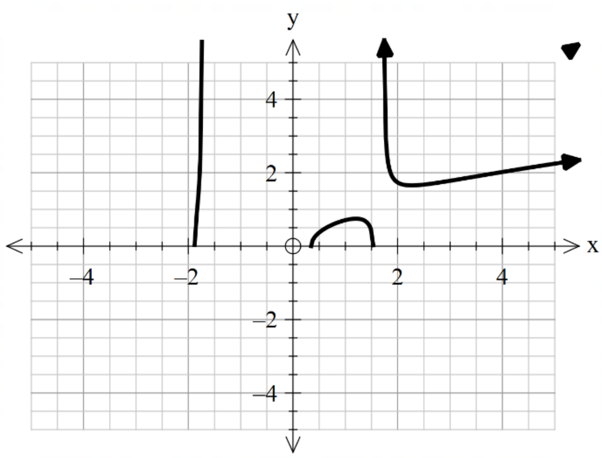


(C)



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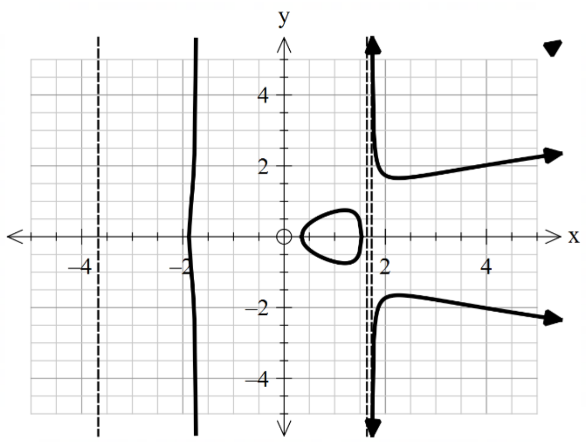

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(D) $\dfrac{(x-5)^2}{9} + \dfrac{(y-4)^2}{16} = 1$

7 The solution of the differential equation $x^3 + y^3\frac{dy}{dx} = 4$

(A) $x^3 + y^3 = 4x + c$

(B) $x^3 + y^3 = 16x + c$

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(B) $$-\frac{1}{f^{-2}(x)}$$

(C) $$f'(x)f(f^{-1}(x))$$

(D) $$\frac{1}{f^{-1}\left(f'(x)\right)}$$

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60 marks

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i) Find $\underset{\sim}{a} + \underset{\sim}{b}$ in component form. **1**

$$\underset{\sim}{a} + \underset{\sim}{b} = (-2+3)\underset{\sim}{i} + (-5-1)\underset{\sim}{j}$$
$$= \underset{\sim}{i} - 6\underset{\sim}{j}$$

ii) Calculate $|-2\underset{\sim}{a} - 7\underset{\sim}{b}|$ **1**

$$\left|-2\underset{\sim}{a} - 7\underset{\sim}{b}\right| = \sqrt{(4-21)^2 + (10+7)^2}$$
$$= \sqrt{578}$$
$$= 17\sqrt{2}$$

(b) A square metal sheet is being heated such that its side length is increasing at a uniform rate of 0.05cms^{-1}. Find the instantaneous rate of change of its area at the instant when the side length is 3.1cm. **2**

$$\frac{dl}{dt} = 0.05$$

$$A = l^2$$

$$\frac{dA}{dl} = 2l$$

Examination continues on next page

$$\frac{dA}{dt} = \frac{dA}{dl} \times \frac{dl}{dt}$$
$$\frac{dA}{dt} = 0.05 \times 2l$$

When $l = 3.1cm$

$$\frac{dA}{dt} = 0.1 \times 3.1$$
$$= 0.31cm^2s^{-1}$$

(c) According to Mars, 24% of M&M plain candies are blue. Assuming the claimed rate of 24% is correct, use the normal approximation of a binomial to find the probability of randomly selecting 100 M&Ms and getting 27 or more that are blue? **2**

$$\sigma = \frac{\sqrt{npq}}{n}$$
$$= \frac{\sqrt{100 \times 0.24 \times 0.76}}{100}$$
$$= \frac{\sqrt{0.1824}}{10}$$

Find z equivalent to 27

$$z = \frac{10(0.27 - 0.24)}{\sqrt{0.1824}}$$
$$= 0.70243 \ldots$$

Look up in Z table

$$P(z > 0.70243) = 1 - P(z < 0.70243)$$
$$= 1 - 0.52790$$
$$= 0.4721$$

(a) i) Express $2\sqrt{3}\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$ **2**

$$R\cos(x + \alpha) = R\cos x \cos\alpha - R\sin x \sin\alpha$$
$$R^2 = \left(2\sqrt{3}\right)^2 + (2)^2$$
$$R^2 = 16$$
$$R = 4, (R > 0)$$

$$\tan\alpha = \frac{2}{2\sqrt{3}}$$

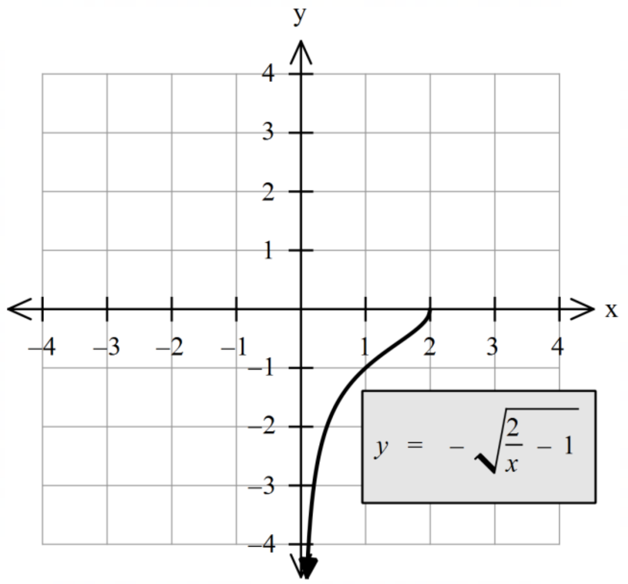
$$\alpha = \tan^{-1}\frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6}$$
$$\therefore 2\sqrt{3}\cos x + 2\sin x = 4\cos(x - \frac{\pi}{6})$$

ii) Hence, solve $2\sin x = 1 - 2\sqrt{3}\cos x,\ 0 \le x \le \pi$ **1**

$$2\sin x = 1 + 2\sqrt{3}\cos x$$
$$2\sin x - 2\sqrt{3}\cos x = 1$$
$$-4\cos(x + \alpha) = 1$$
$$\cos\left(x + \frac{\pi}{6}\right) = -\frac{1}{4}$$

$$x - \frac{\pi}{6} = \cos^{-1}\left(-\frac{1}{4}\right)$$
$$x = \frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{4}\right)$$

(e) The one-to-one function $f(x) = -\sqrt{\frac{2}{x} - 1}$ is graphed below.



(i) State the domain and range of $f(x)$. **1**

Examination continues on next page

$$x\epsilon(0,2], y\epsilon(-\infty, 0]$$

(ii) Sketch the graph of $f^{-1}(x)$. **1**

Graph should:

Pass through (-1,1).

Have 2 as the y-intercept

Come in flat to the y-intercept, as though it was heading to a turning point

(iii) Find the equation of $f^{-1}(x)$ and state its domain and range. **2**

$$x = -\sqrt{\frac{2}{y} - 1}$$
$$x^2 = \frac{2}{y} - 1$$
$$\frac{2}{y} = x^2 + 1$$
$$\frac{y}{2} = \frac{1}{x^2 + 1}$$
$$y = \frac{2}{x^2 + 1}$$
$$x\epsilon(-\infty, 0], \qquad y\epsilon(0,2]$$

(f) $OABC$ is a rhombus, where O is the origin, $\overrightarrow{OA} = \binom{3}{7}$, $\overrightarrow{OC} = \binom{7}{3}$. Find the lengths of the diagonals **2**

$$\left|\overrightarrow{AC}\right| = \left|-\vec{O}A + \overrightarrow{OC}\right|$$
$$= \sqrt{(4)^2 + 4^2}$$
$$= 4\sqrt{2}$$

$$\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA} + \vec{O}C\right|$$
$$= \sqrt{10^2 + 10^2}$$
$$= 10\sqrt{2}$$

Question 12 (15 marks)

(a) Simplify **2**

$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}$$

$$\begin{aligned} &\frac{2\sin x\cos x}{\sin x} - \frac{2\cos^2 x - 1}{\cos x} \\ &= 2\cos x - 2\cos x + \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

(b) Using the dot product find the angle between the vectors given by $\binom{2}{1}$ and $\binom{-2-\sqrt{3}}{-1+2\sqrt{3}}$. **2**

Give your answer correct to two decimal places.

$$\binom{2}{1} \cdot \binom{-2-\sqrt{3}}{-1+2\sqrt{3}} = 2\left(-2-\sqrt{3}\right) + \left(-1+2\sqrt{3}\right) = 2 \times \sqrt{\left(-2+\sqrt{3}\right)^2 + (-1+2\sqrt{3}} \cos\theta$$

$$\begin{aligned} 2\left(-2-\sqrt{3}\right) + \left(-1+2\sqrt{3}\right) &= 2 \times \sqrt{\left(-2+\sqrt{3}\right)^2 + (-1+2\sqrt{3}} \cos\theta \\ -4 - 2\sqrt{3} - 1 + 2\sqrt{3} &= \sqrt{5} \times \sqrt{4 + 4\sqrt{3} + 3 + 1 - 4\sqrt{3} + 12} \cos\theta \\ -5 &= \sqrt{5} \times \sqrt{20}\cos\theta \\ -5 &= 10\cos\theta \\ -\frac{1}{2} &= \cos\theta \\ \theta &= \frac{2\pi}{3} \end{aligned}$$

(c) Using the substitution $u = x^4 + 4x^2$ integrate **3**

$$\int (x^4 + 4x^2 + 5)(x^3 + 2x)dx$$

$$\begin{aligned} u &= x^4 + 4x^2 \\ \frac{du}{dx} &= 4x^3 + 8x \end{aligned}$$

Examination continues on next page

$$\frac{1}{4}du = x^3 + 2x$$

Sub in for $u\ and\ du$

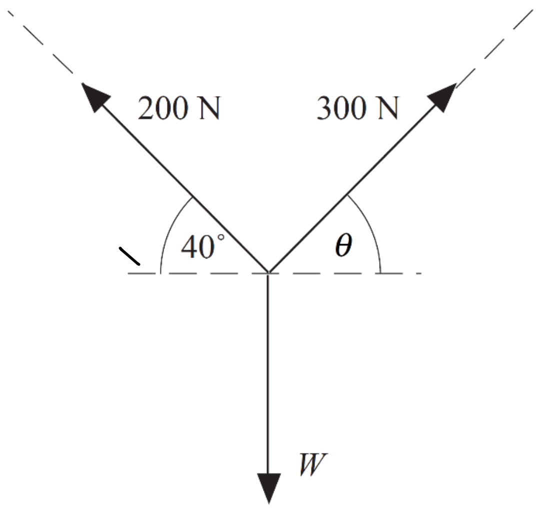
$$\int (u+5)\frac{1}{4}du$$

$$= \frac{u^2}{8} + \frac{5u}{4} + c$$

sub $u = x^4 + 4x^2$

$$= \frac{(x^4+4x^2)^2}{8} + \frac{2(x^4+4x^2)}{4} + c$$

(d) The diagram shows the force of gravity, W, that pulls straight down on an object. Two forces of 200 N and 300 N are exerted by cables as shown. If the resultant force is zero, find θ and W. **3**



Equating up and down and left and right

$$200\sin 40 \ + 300\sin\theta - |W| = 0 \qquad (1)$$

$$300\cos\theta - 200\cos 40 = 0 \qquad (2)$$

$$300\cos\theta = 200\cos 40$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\cos 40\right)$$

$$\theta = 59.28\ldots$$

$$sub\ \theta = 59.28\ldots into\ (1)$$

$$200\sin 40 + 300\sin 59.28\ldots = W$$

$$|W| = -386.49\ldots$$

$$\therefore \theta = 59.28\ldots \text{and } W = -386.49\ldots$$

(e) A bowl of water heated to 100°C is placed in a cool room where the temperature is maintained at $-5\ °C$. After t minutes, the temperature T°C of the water is changing so that $\frac{dT}{dt} = -k(T + 5)$.

i) Show that $T = A\,e^{-kt} - 5$ satisfies this equation, and find the value of A. **2**

$$T = Ae^{-kt} - 5$$
$$\frac{dT}{dt} = -kAe^{-kt}$$
$$T = Ae^{-kt} - 5$$
$$Ae^{-kt} = T + 5$$

sub $Ae^{-kt} = T + 5$ into $\frac{dT}{dt}$

$$\frac{dT}{dt} = -k(T + 5)$$
$$\therefore T = Ae^{-kt} - 5\ satisfiest\ the\ given\ differential\ equation$$

ii) After 20 minutes, the temperature of the water has fallen to 40 °C. How long, to the nearest minute will the water need to be in the cool room before ice begins to form? [that is the temperature falls to 0 °C] **2**

Let $T = 100, and\ \ t = 0$

$$T(0) = A - 5 = 100$$
$$A = 105$$

Let $T = 20\ and\ t = 20$

$$T(20) = 105e^{-20k} - 5 = 40$$
$$e^{-20k} = \frac{45}{105}$$
$$-20k = \ln\frac{45}{105}$$
$$k = \left(\ln\frac{7}{3}\right) \div 20$$

Let $T = 0$

$$T = 105e^{-kt} - 5 = 0$$
$$e^{-kt} = \frac{5}{105}$$
$$-kt = \ln\left(\frac{1}{21}\right)$$
$$t = \ln\left(\frac{1}{21}\right) \div -k$$
$$t = 71.84 \ldots$$

$$\therefore It\ will\ take\ 72 minutes\ for\ ice\ to\ start\ forming$$

Examination continues on next page

(f) A set of encyclopedias is numbered 1 to 13 is placed randomly on a bookshelf. In how many ways can they be placed if encyclopedias 1 and 2 are in the correct order, but not necessarily next to each other. **2**

First place the two encyclopedias 1 and two $\binom{13}{2}$ and then organise the other 11 encyclopedias around these two 11!

$$\binom{13}{2}11! = 3113510400$$

Or

Organise them in any way you would like, then 1 can only be before or after 2 and they are equally likely so

$$\frac{13!}{2} = 3113510400$$

Question 13 (15 marks)

(a) Quality control for the manufacturing of bolts is carried out by taking a random selection of 15 bolts from a batch of 10000. Empirical data has determined that 10% of bolts are defective. If three or more bolts in the sample are found to be defective, that batch is rejected

i) Find the probability that batch is rejected. **2**

As the sample is small, can not use a normal distribution.

$$P(X \geq 3) = 1 - P(X < 3)$$
$$= 1 - P(X = 2) - P(X = 1) - P(X = 0)$$
$$= 1 - \binom{15}{2} 0.1^2 \times 0.9^{13} - \binom{15}{1} 0.1 \times 0.9^{14} - \binom{15}{0} 0.9^{15}$$
$$= 0.184 \ldots$$

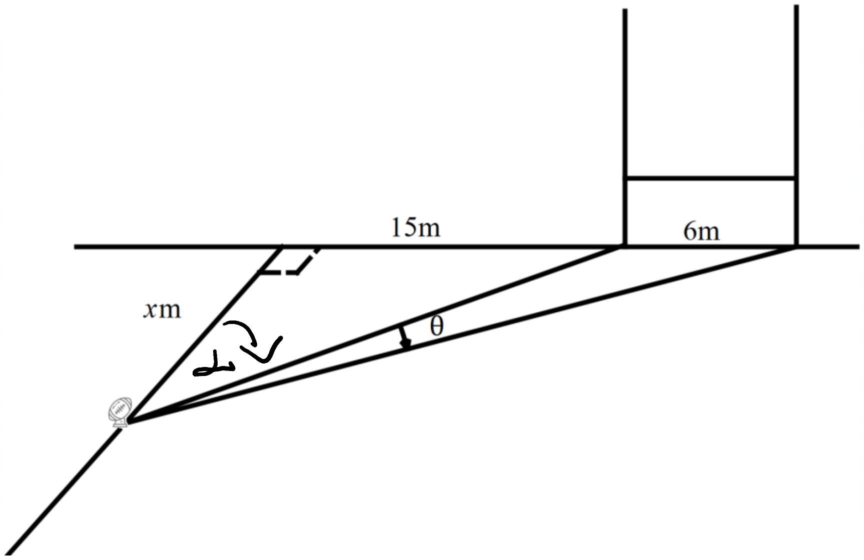
1 mark for setting up the equation, before introducing the binomial expansion.

2 marks final answer

ii) The cost to produce the batch of 10000 is \$20. Each batch is then either sold for \$40 or it is sold as scrap metal for \$5. Find the expected value of the batch of 10000. **1**

$$E(X) = (1 - 0.184 \ldots) * 40 + 0.184 * 5 - 20$$
$$= 13.5578 \ldots$$
$$= \$13.56$$

(b) A rugby player is attempting to kick a ball through the goal posts after his team has scored a try. He can take it anywhere along a line running parallel to the sideline, 15m meters to the side of the goal post. Given that the goal posts are 6m apart:



Examination continues on next page

i) Show that **3**

$$\theta = \tan^{-1}\left(\frac{6x}{315 + x^2}\right)$$

From the diagram

$$\tan(\alpha + \theta) = \frac{21}{x}, \tan\alpha = \frac{15}{x}$$

$$\tan(\alpha + \theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta}$$
$$\frac{21}{x} = \frac{\frac{15}{x} + \tan\theta}{1 - \frac{15}{x}\tan\theta}$$
$$\frac{21}{x} = \frac{15 + x\tan\theta}{x - 15\tan\theta}$$
$$21 - \frac{315}{x}\tan\theta = 15 + x\tan\theta$$
$$6 = \frac{315}{x}\tan\theta + x\tan\theta$$
$$6x = \tan\theta\,(315 + x^2)$$
$$\frac{6x}{315 + x^2} = \tan\theta$$
$$\theta = \tan^{-1}\left(\frac{6x}{315 + x^2}\right)$$

1 mark was for pulling out the two ratios

1 mark for substitution into tan $(\alpha + \theta)$ expansion.

3 marks was for complete and correct working.

ii) Hence find the distance x metres from the try line that makes the value of θ a maximum. **3**

$$f(x) = \frac{6x}{315 + x^2}$$
$$f'(x) = \frac{6(315 + x^2) - 12x^2}{(315 + x^2)^2}$$

$$= \frac{1890 - 6x^2}{(315 + x^2)^2}$$
$$\frac{d\theta}{dx} = \frac{\frac{1890 - 6x^2}{(315 + x^2)^2}}{1 + \left(\frac{6x}{315 + x^2}\right)^2}$$
$$= \frac{1890 - 6x^2}{(315 + x^2)^2 + 6x} = 0$$

$$1890 - 6x^2 = 0$$
$$x^2 = \frac{1890}{6}$$
$$= 315$$
$$x = \pm\sqrt{315} = \pm 3\sqrt{35}$$

x	17	$3\sqrt{35}$	18
$\frac{d\theta}{dx}$	0.000415 ...	0	-0.000128 ...

Therefore $x = 3\sqrt{35}$ will maximise the angle θ

1 mark correct derivative.

1 mark final answer. Should have both positive and negative solutions.

1 mark for showing that the point was a maximum.

(c) Prove by induction that $n^2 + 2n$ is divisible by 8 for all even integers $n \geq 2$ **3**

Show true for $n = 2$

$$2^2 + 2 \times 2 *$$
$$= 8$$
$$\therefore \mathit{true\ for}\ n = 2 *$$

Assume true for $n = k$

$$k^2 + 2k = 8p *$$

Prove true for $n = k + 2$

$$(k+2)^2 + 2(k+2)$$
$$= k^2 + 4k + 4 + 2k + 4$$
$$= k^2 + 2k + 8 + 4k$$
$$= 8p + 8 + 4k\ (\mathit{by\ assumption}) *$$
$$= 8\left(p + 1 + \frac{k}{2}\right), \mathit{Where}\ \frac{k}{2}\ \mathit{is\ an\ integer\ as\ k\ is\ even}$$
$$\therefore k\ \mathit{implies}\ k + 2 *$$
$$\therefore \mathit{By\ the\ principle\ of\ mathematical\ induction}$$
$$n^2 + 2n\ \mathit{is\ divisible\ by}\ 8\ \mathit{for\ all\ even}\ n \geq 2 *$$

NB: Setting n=2k and n=2k+2 made calculations make more intuitive sense

Examination continues on next page

Need all starred lines for 3 marks

Need at least 4 ticks for 2 marks

Need at least two ticks for 1 mark

(d) For nonzero constants c and d, the equation $4x^3 - 12x^2 + cx + d = 0$ has three real roots and two of those roots add to zero. Find $\frac{c}{d}$. **3**

Let the roots of the polynomial be $\alpha, -\alpha\ and\ \beta$

$$\alpha - \alpha + \beta = \frac{12}{4}$$
$$\beta = 3$$

Let $x = 3$ in the polynomial

$$4(3)^3 - 12(3)^2 + 3c + d = 0$$
$$3c + d = 0$$
$$3c = -d$$
$$\frac{c}{d} = -\frac{1}{3}$$

1-mark solving for β

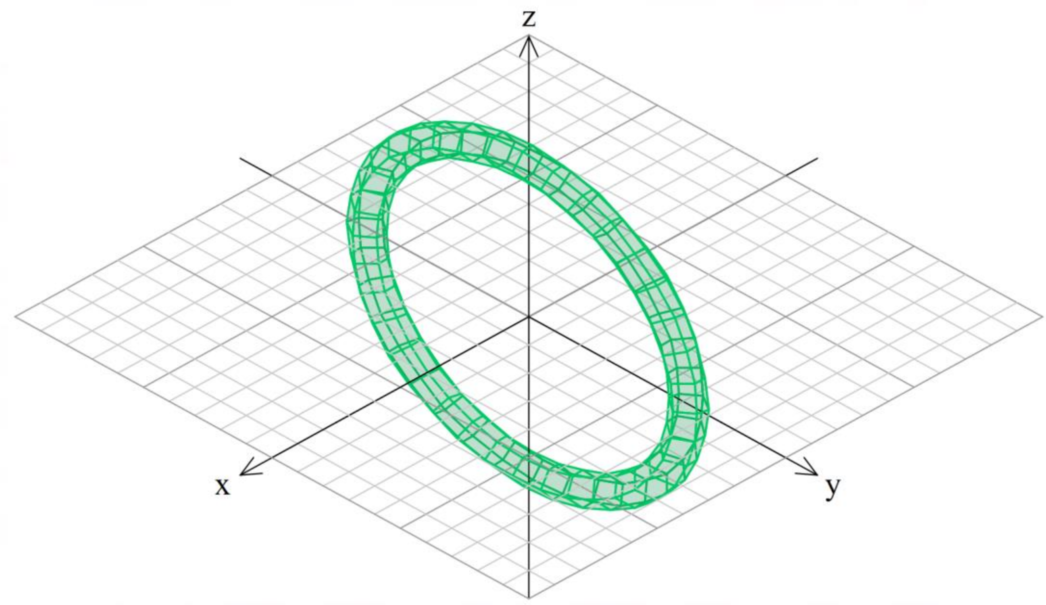
1-mark substituting β into the polynomial

Full marks correct answer from correct working.

Question 14 (14 marks)

(a) A donut shaped pool toy is designed by rotating the circle $(y-125)^2 + x^2 = 144$ around the x-axis. **4**

Find the exact volume of the pool toy.



$$(y-125)^2 + x^2 = 144$$
$$y - 125 = \pm\sqrt{144 - x^2}$$
$y = 125 \pm \sqrt{144-x^2}$ ***(plus/minus should be shown, mark was given generously)***

Volume of the pooltoy is volume of the upper semi-circle rotated around the x-axis minus volume of the lower-semi circle rotated around the x-axis

$$V = \pi\left(\int_{-12}^{12}\left(125+\sqrt{144-x^2}\right)^2 dx - \int_{-12}^{12}\left(125-\sqrt{144-x^2}\right)^2 dx\right)$$
$$= \pi\left(\int 125^2 + 250\sqrt{144-x^2} + (144-x^2) - (125^2 - 250\sqrt{144-x^2} + (144-x^2)dx\right)$$
$$= \pi\int_{-12}^{12} 500\sqrt{144-x^2}dx$$
$$= 500\pi(area\ of\ a\ semicircle)$$
$$= 500\pi\left(\frac{\pi(12^2)}{2}\right)$$
$$= 36000\pi^2 \text{units}$$

Examination continues on next page

1 mark for finding y in terms of x

1 mark for correctly setting up volume as an integral

1 mark for simplifying the difference of integrals or equivalent

1 mark for final volume with correct working

<u>Students struggled to set up the integral as a difference of two functions.</u>

<u>Too many were using values along the y-axis, even though the question rotates around the x-axis</u>

(b) Gary is training for a triathlon. Over a 30-day period, he pledges to train at least once per day, and 45 times in all. If he sticks to his pledge show that there will be a period of consecutive days where he trains exactly 14 times. **3**
(Hint: Consider the sets $\{S_1, S_2, S_3, \ldots, S_{30}\}$ and $\{S_1 + 14, S_2 + 14, S_3 + 14, \ldots, S_{30} + 14\}$ where S_i represents the number of training sessions that have been completed by day i.)

In the set $\{S_1, S_2, S_3, \ldots, S_{30}\}$ all elements are distinct as Gary trains at least once every day. The values the elements of this set can take are between 1 and 45, as he will train at least once on the first day and 45 times across the 30 days.
Similarly the set $\{S_1 + 14, S_2 + 14, S_3 + 14, \ldots, S_{30} + 14\}$ all elements are distinct and will range from 15 to 59.
Given that there are 59 possible values across the two sets and there are sixty elements two of the elements must have the same value. As the elements in each set are distinct from each other, there must be at least one element in common between the sets. This indicates that there is at least one period of consecutive days where he trains 14 times.

1 mark for either counting terms in either set or counting the possible values in the sets or noting that the terms in the sets are distinct.

1 mark for applying counting of sets to a pigeonhole concept

1 mark for correct logic and pigeonhole statement

<u>Barely anybody used the hint given. Consider counting the sets given as per the hint. This would at least give partial marks.</u>

(c) A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water.

Let the amount of salt in the tank be represented by S kg and the time be represented by t min.

i) Show that the expression for $\frac{dS}{dt}$ is **1**

$$\frac{dS}{dt} = 0.1 - 0.1S$$

Inflow of salt: 0.1kg/min
Outflow of salt: $\frac{S}{100} \times 10 = 0.1S$

$$\frac{dS}{dt} = 0.1 - 0.1S$$

<u>Must show outflow as concentration × volume out to achieve mark</u>

<u>Many did not understand what they were meant to show in this question. Some simply restated the identity, you will not receive marks for doing this.</u>

ii) Hence, find an expression for the amount of salt, S, in the tank at any time, t. **3**

$$\frac{dS}{dt} = 0.1 - 0.1S$$
$$\frac{dS}{dt} = 0.1(1 - S)$$
$$\int \frac{1}{1 - S} dS = \int 0.1dt$$
$$-\ln|1 - S| = 0.1t + c$$
$$|1 - S| = e^{-0.1t+c}$$
$$1 - S = Ae^{-0.1t}, where\ A \epsilon R$$
$$S = 1 - Ae^{-0.1t}$$
$$Let\ S = 10\ and\ t = 0$$
$$10 = 1 - A$$
$$A = -9$$
$$S = 1 + 9e^{-0.1t}$$

1 mark for separating to integrate

1 mark integrates including absolute value

1 deals with constant to find solution

(d) By substituting $t = \text{a} \sin(u)$ show that **3**

Examination continues on next page

$$\int \sqrt{a^2 - t^2}dt = \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + \frac{1}{2}t\sqrt{a^2 - t^2}$$

$$t = \operatorname{asin}(u)$$
$$\frac{dt}{du} = \operatorname{acos}(u)$$
$$dt = \operatorname{acos}(u)du$$

$$\int \sqrt{a^2 - t^2}dt = \int \sqrt{a^2 - a^2\sin(u)} \times \operatorname{acos}(u)du$$
$$= \int a\sqrt{1 - \sin^2 u} \times \operatorname{acos}(u)du$$
$$= \int a^2\cos(u)\sqrt{\cos^2(u)}\,du$$
$$= a^2\int \cos^2(u)\,du$$
$$= \frac{a^2}{2}\int \cos(2u) + 1du$$
$$= \frac{a^2}{2}\left(\frac{\sin 2u}{2} + u\right)$$

Sub $u = \sin^{-1}\left(\frac{t}{a}\right)$

$$= \frac{a^2}{4}\left(\sin 2\left(\sin^{-1}\left(\frac{t}{a}\right)\right)\right) + \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right)$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + \frac{a^2}{2}\sin\left(\sin^{-1}\left(\frac{t}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{t}{a}\right)\right)$$
$$= \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + \frac{a^2\left(\frac{t}{a}\right)\left(\sqrt{a^2 - t^2}\right)}{2a}$$
$$= \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + \frac{t}{2}\sqrt{a^2 - t^2}$$

1 mark correct u substitution

1 mark integrate and sub to form an expression in terms of t

1 mark simplified to give final expression

End of Examination